Depletion force

Seminar

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May 21, 2008

Abstract

We discuss depletion forces between colloidal particles in a solution of macromolecules. A basic statistical-mechanical calculation of the force is presented and some different approaches for calculating the depletion force for finite concentrations of the small particles is introduced. Finally we also discuss some experimental studies of the depletion interaction.
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1 Introduction

Suspensions of colloidal particles in a solvent include objects of different length scales. The mesoscopic scale of colloidal particles is typically of the order of 100 nm or more, whereas the microscopic scale of solvent molecules or dissolved ions is a fraction of a nanometer. Many real systems also include particles in between these two scales, such as non-adsorbing polymers, micelles, small colloidal particles (e.g., silica), rod-like viruses, proteins or other macromolecules [1, 2].

In a suspension of large and small particles, the pressure of the latter produces an attractive force between the former. This is referred to as the depletion force. Its range is determined by the size of the small particles, and its profile depends on their shape and density. In the following, we will review the main theoretical facts and discuss some experimental studies of the depletion force.

Two parallel plates

The basic properties of this force can be gained by a very simple calculation based on osmotic pressure of ideal solutions and excluded volumes. When the distance \( h \) between two parallel hard plates in a liquid of small hard spheres with radius \( R_s \) is less than the diameter of spheres, \( h < 2R_s \), the spheres are expelled from the gap between the plates as shown in Fig. 1. This depletion, i.e., the absence of spheres in the gap, leads to anisotropy of the local pressure around the plates which induces the attractive depletion force between the plates.

The effect was first predicted theoretically by Asakura and Oosawa [3, 4] who predicted attractive force for \( h < 2R_s \) and zero force for \( h \geq 2R_s \). They calculated the force per unit area between two parallel plates equal to the osmotic pressure of the surrounding liquid:

\[
\frac{F}{A} = -\rho k_B T \Theta(2R_s - h) .
\]  

(1)

Here \( \Theta \) is the Heaviside function and \( k_B \) is the Boltzmann constant. The depletion force depends on the number density \( \rho = N/V \) and absolute temperature \( T \) of the surrounding liquid, which was treated as an ideal gas.

Entropic force

The physical origin of the depletion force is that the overlap of the restricted volumes of the plates increases the volume accessible to small spheres. As a consequence, their entropy \( S \) is increased and because the hard-sphere mixture is an athermal system, their free energy \( F_H = -TS \) decreases, resulting in an attractive interaction. This is a purely entropic force, which means that its properties are not determined by the underlying microscopic interaction such as the van der Waals interaction between like objects. The properties of this force are determined by the tendency of the whole system to increase its entropy as stated by the second law of thermodynamics, which leads to the force which favors clustering of objects immersed in such a suspension [5].

Phase separation

In the case of colloid particles immersed in a liquid of non-adsorbing polymers, depletion force is a possible cause of phase separation [5]. When the concentration of polymers is
Figure 1: When the distance between the plates is larger than the diameter of the small spheres, \( h \geq 2R_S \), the spheres can move into the gap between the plates and there is no force acting on them. But when the distance is small enough, \( h < 2R_S \), the small spheres are expelled from the gap between the plates and the net force acting on plates is equal to the pressure of the surrounding gas of small spheres.

large enough, the depletion force becomes stronger and the colloids can get crowded in some areas of such a suspension. One practical example of such a phase separation was observed in the mixture of milk protein (casein) and amylopectin, which was found to be in agreement with depletion force theory \[6\]. Another case where depletion interaction is responsible for phase separation is when non-adsorbing polymers are added to the tobacco-mosaic virus (TMV), which has approximately the structure of hard rods. For sufficiently high polymer concentrations mixtures of TMV and polymers exhibit liquid-crystalline ordering. This enables the purification of the virus from a plant \[7\].

A detailed insight into depletion forces is important for studying the stability and phase behavior of colloidal dispersions and for the understanding of properties of polymer-colloid mixtures and other self-assembling phenomena in liquid dispersions. It has also been argued by biologists that the depletion force could be responsible for cellular organization \[2\], thus it could be of broader importance.

## 2 Asakura-Oosawa model

Japanese physicists Sho Asakura and Fumio Oosawa were the first to study the depletion interaction in the 1950s \[3, 4\]. They considered four different configurations:

i) Two parallel, large plates immersed in a solution of hard spheres,

ii) Two parallel, large plates immersed in a solution of thin rodlike macromolecules,

iii) Two spherical bodies immersed in a solution of hard spheres, and

iv) Two parallel, large plates immersed in a solution of flexible chain polymer molecules.

Here we focus on case iii with two large hard spheres of diameter \( D = 2R_B \) in a solution of small hard spheres of diameter \( d = 2R_S \).
Around both large spheres there is a region unavailable to small spheres because of the hard-sphere potential. The volume of this area is \( V_E = \pi(D + d)^3/6 \). As large spheres get close enough, that is when the distance between their centers is \( h < (D + d)/2 \), these regions intersect and the excluded volume unavailable to small spheres \( V_E \) is reduced by the overlapping volume as shown in the Fig. 2. \( V_E \) is the volume of two spherical caps, therefore the reduced excluded volume may be written as:

\[
V'_E = V_E - \frac{2\pi l^2}{3} \left[ \frac{3(D + d)}{2} - l \right],
\]

where \( l = (D + d)/2 - h/2 \) is the width of a lens formed by spherical caps.

As a consequence, the small spheres cannot penetrate into the excluded volume between the large spheres and an osmotic force of the solution of small spheres acts on the outer side of each large sphere. Therefore this force is attractive and depends on the osmotic pressure of the solution.

Asakura and Oosawa’s calculations rely on the methods of statistical mechanics as is appropriate for systems with a large number of particles and thus subject to entropic forces. Partition function for the canonical ensemble is written as

\[
Q = C \int e^{-\beta E} d\Gamma,
\]

where total energy \( E = K + V \) is the sum of kinetic and potential terms, and \( C = (N!h^{3N})^{-1} \) and \( \beta = (k_B T)^{-1} \). The link with thermodynamics is obtained by a statistical definition of Helmholtz’s free energy:

\[
F_H = -k_B T \ln Q.
\]

The force between two particles suspended in a solution is:

\[
F = -\left( \frac{\partial F_H}{\partial h} \right) _T,
\]

where \( h \) is the distance between the centers of the large spheres.

The potential energy can be written as the sum of two parts:

\[
V = V_i + V_e,
\]

where \( V_i \) is the interparticle potential energy and \( V_e \) is the potential energy of interaction of small particles with some external field. In the approximation of a very dilute solution,
the small particles are treated as non-interacting, and therefore the interparticle potential energy is neglected: $V_i = 0$. If all small particles are identical, the partition function can be written as:

$$Q = \frac{1}{N!h^{3N}} \int \exp \left( -\beta(T + V_e) \right) d\Gamma = \frac{1}{N!\Lambda^{3N}} \left[ \int_V \exp \left( -\beta V_e(r, h) \right) d^3r \right]^N ,$$

where $N$ is the total number of small spheres, $V$ is the total volume of the solution, and $\Lambda = h/\sqrt{2\pi mk_BT}$ is the de Broglie thermal wavelength, which was produced by integrating the Maxwell distribution over momenta of small particles. $V_e(r, h)$ represents the potential energy of a small sphere generated by its interaction with the large spheres separated by $h$.

If there is no interaction between the small and the large spheres, then $Q$ is equal to the fraction of phase space available to the former:

$$Q = \frac{V_A^N}{N!\Lambda^{3N}} .$$

$V_A$ is the volume that is available for the centers of small spheres and can be obtained by subtracting the excluded volume $V_E$ from the total volume $V$. The free energy is:

$$F_H = -k_BT \ln \left( \frac{V_A^N}{N!\Lambda^{3N}} \right) = F_{id} - Nk_BT \ln \left( \frac{V_A(h)}{V} \right) ,$$

where Stirling’s approximation was used as $N \gg 1$. $F_{id} = -Nk_BT(1 - \ln NA^3/V)$ is the ideal contribution to free energy and is independent of $h$.

Two cases can be distinguished:

i) The separation of the large spheres is big enough so that the small spheres can penetrate in between them, and

ii) The large spheres are very close to each other and the small spheres cannot penetrate between them.

For both cases the excluded volume was already calculated above [Eq. (2)], thus the available volume is given by:

$$V_A = \begin{cases} V - V_E, & h \geq d + D \\ V - V_E + \pi/6(D + d - h)^2(D + d + h/2), & h < d + D \end{cases} .$$

The logarithm in Eq. (7) can be linearized because the macroscopic volume $V$ is much larger than the excluded volume $V_E$. For $h < d + D$ we obtain:

$$\ln \left( \frac{V_A}{V} \right) \approx -\frac{V_E}{V} + \frac{\pi}{6V}(D + d - h)^2(D + d + h/2) .$$

Finally the depletion force reads:

$$F = \begin{cases} 0, & h \geq d + D \\ -\frac{N}{4\pi}k_BT\pi(D + d - h)(D + d + h) = -p_0S, & h < d + D \end{cases} .$$

The negative value of $F$ is a signature of the attractive force between large spheres, so that the depletion force here is always attractive. Because interactions between the small spheres were neglected, the osmotic pressure is expressed by the van’t Hoff’s limit
Figure 3: Depletion force between two large spheres in a suspension of small spheres with size ratio \( s = R_S/R_B = 0.1 \) is drawn with a solid line. The dashed line represents the profile of depletion force between two plates in a suspension of small spheres. As can be seen in this plot, the profile of depletion force between two big spheres is almost linear. The range of the force is in both cases finite and equals to the radius of small spheres.

\[ p_0 = \rho k_B T, \] where \( \rho \) is number density of small spheres. \( S \) is the circular area of the overlapping volume of radius \( r = \sqrt{(D + d)^2 - h^2}/2 \). The force between two large spheres in a suspension of small spheres is presented in Fig. 3.

The characteristics of depletion forces is the same in all cases mentioned above. Their range is equal to the diameter of the small spheres, and their magnitude is the pressure of the ideal gas. The depletion potential is conventionally defined as \( F = -\partial \mathcal{V}_{eff}/\partial h \) and \( \mathcal{V}_{eff}(h = D + d) = 0 \), it is equal to the second term in free energy Eq. (7). Depletion potential as calculated by Asakura-Oosawa theory for different configurations is shown in Fig. 4, where size ratio is smaller than in Fig. 3.

Figure 4: Relative depletion potential as a function of separation as given by Asakura-Oosawa model for two large spheres, two plates and a large sphere and a plate immersed in a fluid of small hard spheres of radius \( R_g = 20 \text{ nm} \). The radius of large sphere is in all cases \( 10^3 \text{ nm} \) and the surface of plates is \( 150 \times 150 \text{ nm} \), \( s = 0.02 \). \( H \) is the distance between the surfaces of large bodies. It can be seen here that the magnitude of depletion potential is smallest for the case of two large spheres and largest in case of two plates [8].
3 More accurate models

In the basic calculation presented in the previous chapter, we have assumed that the local density of the small particles (also called the depletant particles) changes only when \( h < D + d \). Otherwise the density of the surrounding liquid is isotropic, although there are external potentials due to immersed objects present. In reality, liquids are homogeneous in the bulk but in the vicinity of external potentials, for instance near confining walls, inhomogeneities appear. To study these inhomogeneities, one must use more sophisticated theories [5].

The potential energy of interaction between depletant particles with some spatially varying external field has to be included in potential energy [Eq. (4)] of the Hamiltonian. Let us consider a large hard sphere of radius \( R_B \) located at \( r = (2R_B + h)\mathbf{e}_z \) represented by some potential \( \mathcal{V}_2 \). Let there be another large hard sphere (case \( a \)) or a hard planar wall (case \( b \)) located at the origin represented by some potential \( \mathcal{V}_1 \). Both obstacles are surrounded by a fluid of small particles of radius \( R_S \). The external potential acting upon the small particles is therefore:

\[
\mathcal{V}_e(r, h) = \mathcal{V}_1(r) + \mathcal{V}_2\left(r - (2R_B + h)e_z\right),
\]

where \( h > 0 \) is the distance between the obstacle and the sphere and \( \mathcal{V}_2 \) is a hard-sphere potential \( \mathcal{V}_2(r) = \delta(\Theta(r - (r_b + r_s))) \), here \( \delta \) is Dirac delta function and \( \Theta \) is Heaviside function. In case \( a \), when there are two large spheres, \( \mathcal{V}_2(r) = \mathcal{V}_1(r) \). This configuration is illustrated in Fig. 5. The consequence of the presence of external potential is nonuniformity of local density \( \rho(r) \) and thus the anisotropy of the pressure exerted on the big spheres.

In the previous chapter, the effect of excluded volumes of the small particles themselves was not taken in account, which leads to a linear correction in virial expansion of pressure. Our result was thus a first-order approximation for local density and therefore gives us the pressure of an ideal gas.

Figure 5: Two big spheres at distance \( h \) apart represent the external potential acting upon a fluid of small spheres. The presence of this potential causes the anisotropy of local pressure of small spheres resulting in attractive force between the big spheres [5].
4 Density functional theory

The local particle density is defined by:

\[ \rho(\mathbf{r}) = \left\langle \sum_{i=1}^{N} \delta(\mathbf{r} - \mathbf{r}_i) \right\rangle, \tag{12} \]

where the angular brackets denote the average over the grand canonical ensemble, if we consider an open system where exchange of particles with a reservoir is allowed. In a homogeneous fluid the local density reduces to the bulk density \( \rho = N/V \), which is uniquely determined by the chemical potential of the particles \( \mu = (\partial F/\partial N)_{T,V} \).

When external potential is present, the local density \( \rho(\mathbf{r}) \) becomes non-uniform. Since the probability distribution for a grand canonical ensemble is proportional to \( \exp\left(-\beta(K + V_i + V_e(\mathbf{r}) - \mu)\right) \), it can be seen from the definition of local density [Eq. (12)] that \( \rho(\mathbf{r}) \) is uniquely determined by \( \psi(\mathbf{r}) = \mu - V_e(\mathbf{r}) \). In the absence of \( V \), the grand potential \( \Omega \) is a function of \( \mu \), but if the external field is present it becomes a functional of a modified potential \( \psi : \Omega = \Omega[\psi(\mathbf{r})] \). Because the local density \( \rho(\mathbf{r}) \) is more practical thermodynamic variable than \( \psi(\mathbf{r}) \), the variables are changed via a generalized Legendre transformation:

\[ \Omega[\psi] = F_H[\rho] - \int \psi(\mathbf{r}) \rho(\mathbf{r}) d^3r. \tag{13} \]

\( \Omega \) is then minimized according to the variational principle.

The basic theorems of density functional theory\footnote{The two key results of the density functional theory are called Hohenberg-Kohn-Mermin theorems. The proof can be found in Ref. \cite{10}, p. 395.} state that for any fluid exposed to an arbitrary external potential \( V_e(\mathbf{r}) \), the intrinsic free energy is a unique functional of equilibrium single-particle density \( \rho(\mathbf{r}) \). Thus any quantity dependent on free energy is also a unique quantity of \( \rho(\mathbf{r}) \), meaning that all measurable equilibrium quantities are unique functionals of the number density \( \rho(\mathbf{r}) \).

The exact result within density functional theory for the depletion force in the presence of potential Eq. (11) is \cite{5}:

\[ \beta F = - \int_S \rho(\mathbf{r}) \mathbf{n} dA, \tag{14} \]

where \( S \) is a surface of a sphere with radius \( R_B + R_S \) centered at center of sphere 2 and \( \mathbf{n} \) is the normal unit vector perpendicular to this sphere pointing outwards. \( \rho(\mathbf{r})/\beta \) can be interpreted as local kinetic pressure, and for a hard sphere the net force is a surface integral of this pressure. If there is no external potential \( V_1 = 0 \), then \( \rho(\mathbf{r}) \) is radially symmetric and there is no net force. But if \( V_1 \) is nonzero, \( \rho(\mathbf{r}) \) is nonsymmetric and leads to a nonzero force \( F_z(h) \) \cite{5}.

The derivation of this formula is valid for any fluid, not only for hard spheres, and for any external potential \( V_1 \). Eq. (14) can also be obtained by introducing mean forces acting on the large particles \cite{9}.

For a radially symmetric external potential \( V_1 \) (let \( \mathbf{e}_z \) be the symmetry axis), the local density has the same symmetry and Eq. (14) can be written as \cite{5}:

\[ \beta F_z(h) = \pi \epsilon \int_h^\infty \left[ (R_B + R_S) - \epsilon(L-h)/2 \right] \Delta \rho(L) dL. \tag{15} \]

This equation is exact and \( \epsilon \) is a geometrical factor, which has a value 1 for \( V_1 \) representing the potential of another hard sphere (case a) or 2 for the case of a hard wall at the origin.
(case b) [5]. \( \Delta \rho(r) \) is the difference of the contact densities between the left and the right hemisphere of sphere 2.

The calculation of the contact density \( \rho(r) \) can be done numerically by molecular dynamics simulation or by minimizing a suitable density functional. This is still a very tough numerical task, so usually approximations are used. Different approximations for the density profile of the fluid at contact with the sphere lead to different approximations for the depletion force. We will discuss some of the approximations. These approximations help us to physically understand the entropic attraction and repulsion.

If \( \Delta \rho(L) \) is a quickly decaying function of \( L \), then the second term in the integrand in Eq. (15) is negligible:

\[
\beta F_z(h) = \pi \epsilon \rho_s \Theta(2R_S - h)(R_B + R_S)(2R_S - h) .
\] (16)

This approximation is called the projection approximation because the projection onto \( z \) axis is not taken into account. If \( \Delta \rho(L) \) for a hard sphere and a wall are similar, it then follows from this approximation that the ratio of depletion force in case b and depletion force in case a is around 2, which can also be seen in Fig. 4.

Another example of approximation is to first calculate the density profile \( \rho(r) \) for a system with \( V_1 \) only and then for system with \( V_2 \) only. The density profile of the system containing both potentials is then estimated by superposing these densities [5].

5 Extension of Asakura-Oosawa model to finite densities

In the Asakura-Oosawa theory presented in chapter 2, it was assumed that the local density changes only when \( h < d \) when the small spheres cannot penetrate into the excluded volume between the large objects. This means that the local density between objects is 0 and the local density on the other hemisphere is the density of the bulk liquid. The contact density is thus:

\[
\Delta \rho(L) = \begin{cases} 
0, & L > 2R_S \\
-\rho_s, & L < 2R_S 
\end{cases} .
\] (17)

By inserting this contact density difference into Eq. (15), we would obtain the result of Asakura and Oosawa Eq. (10). By using the projection approximation, we obtain:

\[
\beta F_z(h) = \pi \epsilon \rho_s \Theta(2R_S - h)(R_B + R_S)(2R_S - h) .
\] (18)

For small size ratios \( s = R_S/R_B \) the ratio of forces given by Eqs. (10) and (18) approaches 1, for \( s = 0.1 \) the ratio is 0.91 for the case of two spheres. Projection approximation thus becomes good in this limit [5].

Virial expansion

At large densities simple fluids made of hard spheres or hard rods (liquid crystals) are dominated by the excluded-volume effects. In the bulk, the equation of state can be expanded using the virial theorem of classical statistical mechanics. To first order, the virial expansion coincides with the excluded-volume theory where each small hard-sphere particle is assigned an excluded volume equal to eight times its own volume \( v_E = 4\pi\sigma^3/3 \).
into which the centers of other particles can not be placed. At very small densities, the total available volume of the system is \( V_A = V - NvE \). From Eq. (6) and \( p = -\frac{\partial F_H}{\partial V} \) and by using the Stirling’s formula, \( \ln N! \approx N \ln N - N \), we find [10]:

\[
p = \frac{N}{V} k_B T \frac{1}{1 - 4\eta} \approx \frac{\rho}{\beta} \left[ 1 + 4\eta + \mathcal{O}(\eta^2) \right].
\] (19)

The last approximation is justified since the available volume is only valid for very low packing fractions \( \eta = \frac{\pi \sigma^3}{6v} \ll 1 \). Here \( \sigma = 2R_S \) is diameter of small particles and \( v = V/N \).

This result is just a first-order correction to the ideal gas law in virial expansion of \( p \). At larger densities the calculation of available volume is much more difficult because excluded volumes of each particle are overlapping. One of equations of state for larger densities is provided by scaled particle theory, which gives the approximate result for the work required to create a spherical cavity in the fluid of hard spheres, which is the same as inserting another hard-sphere particle of the same size into such a liquid. This reversible work is then related to the chemical potential and to the pressure via elementary thermodynamic relations [1, 10]. The resulting equation of state is:

\[
\frac{\beta p}{\rho} = \frac{1 + \eta + \eta^2}{(1 - \eta)^3}.
\] (20)

This equation of state is very accurate even for packing fractions near freezing.

Even more accurate is the Carnahan-Starling equation of state, which is a semi-empirical equation of state for hard spheres. It was derived using an infinite series that analytically approximated the first six known virial coefficients [1]:

\[
\frac{\beta p}{\rho} = 1 + 4\eta + 10\eta^2 + 18.365\eta^3 + \cdots \approx 1 + \sum_{i=1}^{\infty} \frac{(i^2 + 3i)\eta^i}{(1 - \eta)^3}.
\] (21)

This is the correction of scaled particle theory and is in excellent agreement with computer simulations over the entire fluid range [Fig. 6]. The error does not exceed 1% [10].

![Figure 6: Comparison of pressure given by Carnahan-Starling equation of state, scaled particle theory and ideal gas law. The pressure given by Carnahan-Starling equation is very exact over the entire fluid range.](image)
Improved Asakura-Oosawa model

To improve the Asakura-Oosawa model, the projection approximation is used [Eq. (18)] and \( \rho_s \) is replaced by \( \rho' \) from [5]:

\[
\frac{\rho'}{\rho} = 1 - \frac{\eta}{2} + \frac{3}{2} + \frac{\eta(1-\eta)2R_B - 2R_S}{(1-\eta)^3} \frac{2R_B + 2R_S}{R_B - 2R_S} .
\]

(22)

\( \rho' \) is the contact density for the fluid of small spheres when one large hard sphere represented by \( V_2 \) is present and the other obstacle, represented by \( V_1 \), is ignored. It was obtained by fitting simulation data [5]. In the case of a planar wall \( R_B \to \infty \), \( \rho' \) reduces to Carnahan-Starling contact density [Eq. (21)].

This improvement of Asakura-Oosawa model does not tell us anything about the repulsive and oscillatory depletion force, but it gives a good approximation for high packing fractions in basic attractive regime [5].

6 Derjaguin approximation

The interaction energy between two planar surfaces can be related to the force between two curved surfaces like two spherical bodies or between a spherical body and a planar wall. This is done using geometrical approximations, first proposed by Derjaguin in 1934. The Derjaguin approximation is valid for any type of a force law [11].

Let there be two spheres with radii \( R_1 \) and \( R_2 \) with centers lying on \( z \) axis at distance \( h + R_1 + R_2 \) apart. The relation between the force in \( z \) direction \( F \) acting between two spheres and the interaction potential per unit area \( W(h) = \int_h^\infty f(z) \, dz \) of two planar surfaces at distance \( h \) apart if \( h \ll R_1, R_2 \) is:

\[
F(h) \approx 2\pi \frac{R_1R_2}{R_1 + R_2} W(h) ,
\]

(23)

where \( f(z) \) is the normal force per unit area between two flat surfaces at distance \( z \) apart [Fig. 7]. The force between two hard planes in a solution of much smaller hard spheres can thus be used to calculate the force between two large colloidal particles in the same solution.

In the case of depletion force, the local density of small spheres changes because of the presence of external potential due to two large spheres whose centers are \( h + 2R_B \) apart. For small size ratios \( s \ll 1 \), this external potential can be approximated by the potential of two planar walls separated by \( h \). Thus the local density at the left hemisphere of hard sphere 2 [Fig. 5] is substituted with the local density of the liquid confined to a slit of width \( h \) and the local density on the right hemisphere is approximated by the local density at a single planar wall (the same as a slit of infinite width). The difference in contact densities is thus \( \Delta \rho(h) = \rho_{slit}(h) - \rho_{slit}(\infty) \). By inserting this in Eq. (16) and setting \( \epsilon = 1 \) the same result is obtained as by the Derjaguin approximation [Eq. (23)] in case \( R_2 \gg R_1 \), which is true for a sphere near a flat wall.

When \( 0 < h < 2R_S \), all small spheres are squeezed out of the slit and the contact density in the slit is 0. The force is then given by the bulk pressure \( p(\rho) = -k_BT\Delta \rho(h) \). If \( \gamma(\rho, h) \) is the contribution to the grand potential defined by \( -p(\rho) = -\partial \gamma(\rho, h)/\partial h \), then the depletion force in Derjaguin approximation is given by [3]:

\[
F(h) = -\pi \epsilon (R_B + R_S)[p(\rho)(2R_S - h) + \gamma(\rho, \infty)] , \quad 0 < h < 2R_S .
\]

(24)
The Derjaguin approximation relates the force between two spheres to the force between two flat surfaces. This is done by integrating the force between small circular regions on one surface and the opposite surface, which is assumed to be locally flat. It can be used for any type of force law in case the separation $D \ll R_1, R_2$ and is very useful, because it is usually easier to derive a force between two planar surfaces. Here $\gamma(\rho) = \frac{1}{2} \gamma(\rho, \infty)$ is the interfacial tension at the wall-fluid interface and is negative for hard spheres at the wall. It is independent of $h$ because the Derjaguin approximation neglects the finite curvatures effects. The last term in the above equation is thus repulsive, we can expect the depletion force to be repulsive at separation $h = 2R_S$. This is in contrast with the Asakura-Oosawa model and the virial expansion corrections where depletion forces are purely attractive.

The contact density difference between two spherical bodies in a liquid of small spheres can thus be replaced by the contact density difference of two planar walls at same distance apart, which is easier to calculate. The Derjaguin approximation can be used for any type of particles that are small enough in comparison with large spheres. The approximation is good when the large spheres are significantly larger than the depletant particles and becomes exact when radius of large spheres $R_B \rightarrow \infty$. It is valid for all fluid densities $\rho$, whereas the Asakura-Oosawa model is valid only in very small densities.

The usefulness of the Derjaguin approximation lies in the fact that highly accurate, analytic formulas for $p(\rho)$ and $\gamma(\rho, \infty)$ are known for the case of two planes. For example, the above mentioned Carnahan-Starling equation for the pressure and scaled particle theory result for $\gamma(\rho)$ and many more sophisticated results are known.

### 7 Integral equations theory

The depletion potential is a part of potential of mean force $V(r)$ between a pair of large particles in the presence of small particles which can be determined by the radial distribution function $g(r)$ via relation:

$$V(r) = -k_B T \ln g(r) .$$

In order to obtain radial distribution function, the total correlation function $h(r)$ is calculated. It is related to $g(r)$ by:

$$h(r) = g(r) - 1 .$$
A possible approach to calculate $h(r)$ is provided by Ornstein-Zernike theory, which states that the total correlation function between particle 1 and particle 2, consists of two parts. The first part is the direct correlation between these two particles, $c(r)$, and the second part consists of all possible indirect correlations through the other particles in the system. The second part is expressed as the convolution of $c(r)$ and $h(r)$. This relation is called Ornstein-Zernike equation and can be written for particles of type $i$ and $j$ as

$$h_{ij}(r_{12}) = c_{ij}(r_{12}) + \sum_{k=1}^{N} \rho_k \int h_{jk}(r_{13}) c_{kj}(r_{32}) dr_3 , \tag{27}$$

where $N$ is the number of different types of particles in a system and $\rho_k$ is the number density of particle of type $k$. In order to solve Ornstein-Zernike equation, a closure relation between $h(r)$ and $c(r)$ has to be used which leads to an integral equation for $g(r)$. Several such closure relations exist and one of the most commonly used for a mixture of hard spheres is Percus-Yevick relation:

$$h_{12}(r) + 1 = \exp(-\beta V_{12}(r))[1 + h_{12}(r) - c_{12}(r)] , \tag{28}$$

where $V_{12}(r)$ is the pair potential between particle 1 and particle 2 (interparticle potential).

Analytical solution for one-component hard-sphere system exists, but for more-component solutions and polydisperse models, the Ornstein-Zernike equation with appropriate closure relation has to be solved numerically [9, 12].

The results obtained within this theory are represented and compared with experimental results and computer simulations in Figs. 8 and 9. Both figures show a repulsive peak and at higher concentrations another attractive minimum is observed, both of which are not predicted by Asakura-Oosawa model. The repulsive peak appears at the separation of around one small sphere diameter and is also seen in experimental results in Fig. 11. This repulsion is qualitatively in accordance with Derjaguin approximation result [Eq. 24].

![Figure 8: Depletion force between two mica cylinders in a solution of micelles at $\eta = 0.29$. The force oscillates with the period of about the diameter of micelles 9 nm. Integral equation theory for monodisperse hard-sphere depletant particles is not in agreement with the measurements, but when the Gaussian distribution for the size of particles is used in integral equation theory, the agreement becomes better [12].](image-url)
At still larger densities the depletion force becomes oscillatory with the wavelength of the order of the small particle diameter. Oscillations appear because at large densities, small spheres tend to form layers around the large objects as shown in Fig. 10. When the gap between the plates is an integer multiple of diameter of small spheres, the free energy is smaller than at separations incommensurate with the small sphere diameter [13].

In Fig. 9 results are in agreement with simulation results even at large distances, thus the layering of small particles is well described within integral equations theory. Fig. 8 also shows results for the depletion force where the depletant particles’ sizes were Gaussian distributed. A comparison of theoretical predictions for the depletion force with the experimental measurements show that they are in good agreement.

Figure 9: The depletion force between two large spheres in a fluid of small spheres as measured by molecular dynamics simulation (empty circles) for size ratio \( s = 0.1 \) and packing fraction \( \eta = 0.31 \). The simulation results are compared with integral equation theory for the radial distribution function where the dashed line represents Percus-Yevick closure relation and a solid line the closure relation of Rogers and Young. The agreement with simulation results is good at large distances [9].

Figure 10: At larger densities of the depletant particles, the latter tend to form layers around large objects. When the gap between the plates is commensurate to the diameter of small spheres, the free energy is smaller than at separations incommensurate with the small sphere diameter. This causes the depletion force to oscillate [13].
Computer simulations

There are two distinct methods used in computer simulations of large ensembles of particles: Molecular dynamics and Monte-Carlo method. In molecular dynamics simulation, the motion of particles in a system with initially defined conditions is tracked by numerically integrating the classical equations of motion given by the known interparticle potential. The equilibrium properties are then obtained by the time-averaging over the dynamical history of the system. This simulation is deterministic in contrast to Monte-Carlo simulation, where particles are being randomly displaced and the next state of a particle is chosen such that it is allowed by interparticle potential and boundary conditions and also according to the probability density of the particular statistical ensemble. The ensemble average of some quantity is then obtained by unweighted averaging over the resulting set of phase states [1, 10].

Computer simulations provide a very accurate description of such systems, but the results depend on how well-defined models were used. Hard spheres are well-defined and the results are qualitatively in accordance with theoretical predictions. The molecular dynamics simulation of large hard spheres in a solution of small hard spheres for the diameter size ratio $s = 0.1$ and for packing fraction of small spheres $\eta = 0.31$ is shown in Fig. 9.

8 Experimental results

Measuring the depletion force is not an easy task, because this force is usually extremely weak and there are also many other forces present between colloidal particles and polymer particles like attractive van der Waals forces, electric repulsion due to Debye double layers in ionic solutions, and hydrodynamic effects. Therefore the measuring device has to be very precise and the experimental setup has to exclude other interactions. It is also very difficult to control the positions of colloidal particles. Another problem is to minimize polydispersity of colloidal particles; all particles have to be of the same size as they are treated in theoretical predictions where size polydispersity is usually neglected.

The first unambiguous measurement of depletion force was conducted by Evans and Needham in 1988 [14], who measured the depletion attraction between two bilayer membranes in a solution of macromolecules and found it to be the same to that of osmotic pressure of the surrounding liquid. Here we will present some more recent experiments.

Line optical tweezer and digital video microscopy

Depletion interaction between colloidal spheres in a solution of small spheres can be measured using a line optical tweezer which uses optical gradient forces to trap a pair of large spheres [13]. The spheres then behave as if they were constrained to a frictionless rod of light, where they undergo Brownian diffusion. Digital video microscopy is used to measure the probability of finding the two spheres at a given separation $r$, which is related to the Helmholtz free energy by $F_H(r) \propto \exp \left( -\frac{F_H(r)}{k_B T} \right)$. The large spheres were polymethylmethacrylate of diameter 1100 ± 15 nm and the small were polystyrene spheres of diameter 83 nm. Salt, which works as a buffer, and SDS surfactant (sodium dodecyl sulfate) was also added to prevent aggregation of colloids due to strong attractive van der Waals forces.

The volume fraction $\phi_S$ of the small spheres was varied from 0.04 to 0.42, which is very close to point of crystallization. Interactions between small spheres are expected to be
screened Coloumbic repulsion with a screening length of 3 nm. This is small comparing to their radius, thus interaction is very near to that of hard spheres. The measured interaction potential as a function of the distance between large particles $r$ for various packing fractions $\phi_S$ is shown in Fig. [11].

![Depletion interaction potential as a function of separation $r$ between the centers of two large spheres in a solution of small spheres as measured by line optical tweezer experiment.](image)

Figure 11: Depletion interaction potential as a function of separation $r$ between the centers of two large spheres in a solution of small spheres as measured by line optical tweezer experiment. Each curve represents a result for a different packing fraction $\phi_S$, the curves are aligned horizontally for a clearer representation. The units of potential energy are in $k_B T$, which is a typical order of magnitude for a depletion interaction. Strong attraction at short range is observed and repulsive interaction at separation of about one small sphere diameter for $\phi_S > 0.1$. At larger $\phi_S$ the depletion force becomes oscillatory due to the tendency of small spheres to form layers around the large spheres. The schematic representation of arrangement of small spheres for three different regimes of the depletion force is shown in cross section on the right side [13].

The results of this measurement show that there is a strong attraction at short range, which is well explained by Asakura-Oosawa model. At packing fractions $\phi_S > 0.1$, the depletion potential is repulsive for separations of around one small sphere diameter from contact and for even higher packing fractions $\phi_S > 0.25$, the depletion potential becomes oscillatory. Both phenomena are not expected by the Asakura-Oosawa model.

**Laser radiation pressure measurement**

Here the pressure of laser beam is exerted on a polystyrene latex colloid particle with diameter 1.0 $\mu$m near a hard wall in a suspension of non-adsorbing polymers - PEO
(polyethylene oxide) with the radius of gyration $R_g = 0.101 \mu m$ in water. The schematic representation of experiment is shown in Fig. 12. The particle was trapped in an optical cell made of micro slide glass. The minimum laser intensity that is required to blow the particle away from the wall is then measured. The results for the depletion force as a function of polymer concentration are shown in Fig. 12 (right), where the solid line is the least square fit for the measured vertical lines (100 measurements for each concentration) and the dotted line represents the result from the Asakura-Oosawa model. The vertical dotted line is the maximum polymer concentration where the depletion force is proportional to polymer concentration.

In this case the difference between the experiments and theory can be attributed to many factors like the error in estimation of depletion region in Asakura-Oosawa equation, which is not always equal to radius of gyration of polymers $R_g$ or repulsion between glass walls and polymers, which causes an increase of depletion force or even physisorption of polymers to latex particles. Discrepancies are discussed in more detail in Ref. [15]. The decrease of force beyond the concentration $C^\dagger$ is due to the softness of polymers, which start to overlap after a certain concentration and their correlation length starts to decrease. Despite all this factors, the observed depletion force is almost the same as the theoretical prediction.

**Atomic force microscopy**

Depletion forces can also be directly measured using atomic force microscopes with colloidal probes, where a micrometer-sized colloidal particle is attached to an AFM cantilever. The basic principle is to measure the force which drags a cantilever with a colloidal probe to a silica surface immersed in a fluid. The deflection of the cantilever is then measured with laser reflection on it and because the spring constant of the cantilever is known, force can be computed [16]. Here silica surface was immersed in a supramolecular polymer (EHUT/DBUT) solution of concentration 100 g/L in cyclohexane (nonpolar solvent).
The scheme of experiment and its results are shown in Fig. 13 where the solid line represents a fit to Asakura-Oosawa model.

Figure 13: A scheme of atomic force microscope with colloidal probe (left) where the depletion force between the colloidal particle on a probe and the planar surface was measured. The results for a large number of measurements (right) where the solid line represents a linear fit show that a relatively large experimental error was present. The range and magnitude of the depletion force are in agreement with predictions [16].

9 Conclusions

The depletion force is well understood both theoretically and experimentally. Although all characteristics of real-world systems cannot be embedded in theoretical models, the results show good agreement with experiments and simulations.

Studying depletion forces is important because it acts between particles of various shapes and in different types of dispersions. It is thus present in mixtures of colloids and polymers as well as in environments crowded with macromolecules, such as a cell. It has also been shown that it is strong enough to cause the formation of colloidal crystals or floculation if the colloidal suspension is mixed with the high enough concentration of polymers.
References


